Geometric Inverse Problems

With Emphasis on Two Dimensions

GABRIEL P. PATERNAIN
University of Cambridge

MIKKO SALO University of Jyväskylä, Finland

GUNTHER UHLMANN
University of Washington



Contents

	Foreword by András Vasy Preface		<i>page</i> xi	
			xv	
	Acknowledgements		xxiii	
1	The Radon Transform in the Plane		1	
	1.1	Uniqueness and Stability	1	
	1.2	Range and Support Theorems	5	
	1.3	The Normal Operator and Singularities	8	
	1.4	The Funk Transform	19	
2	Radial Sound Speeds		25	
	2.1	Geodesics of a Radial Sound Speed	25	
	2.2	Travel Time Tomography	31	
	2.3	Geodesics of a Rotationally Symmetric Metric	36	
	2.4	Geodesic X-ray Transform	39	
	2.5	Examples and Counterexamples	45	
3	Geometric Preliminaries		52	
	3.1	Non-trapping and Strict Convexity	52	
	3.2	Regularity of the Exit Time	58	
	3.3	The Geodesic Flow and the Scattering Relation	63	
	3.4	Complex Structure	65	
	3.5	The Unit Circle Bundle of a Surface	75	
	3.6	The Unit Sphere Bundle in Higher Dimensions	81	
	3.7	Conjugate Points and Morse Theory	87	
	3.8	Simple Manifolds	99	
4	The Geodesic X-ray Transform		107	
	4.1	The Geodesic X-ray Transform	107	
	4.2	Transport Equations	110	

viii Contents

	4.2	Pestov Identity	112
	4.3	Injectivity of the Geodesic X-ray Transform	114
	4.4	Stability Estimate in Non-positive Curvature	117
	4.5 4.6	Stability Estimate in the Simple Case	123
	4.0 4.7	The Higher Dimensional Case	126
			120
5	Regularity Results for the Transport Equation		130
	5.1	Smooth First Integrals	130
	5.2	Folds and the Scattering Relation	132
	5.3	A General Regularity Result	135
	5.4	The Adjoint $I_{\mathcal{A}}^*$	138
6	Verti	ical Fourier Analysis	142
•	6.1	Vertical Fourier Expansions	142
	6.2	The Fibrewise Hilbert Transform	149
	6.3	Symmetric Tensors as Functions on SM	151
	6.4	The X-ray Transform on Tensors	157
	6.5	Guillemin-Kazhdan Identity	162
	6.6	The Higher Dimensional Case	166
7	The X-ray Transform in Non-positive Curvature		171
	7.1	Tensor Tomography	171
	7.2	Stability for Functions	173
	7.3	Stability for Tensors	179
	7.4	Carleman Estimates	182
	7.5	The Higher Dimensional Case	186
8	Micr	colocal Aspects, Surjectivity of I_0^*	189
	8.1	The Normal Operator	189
	8.2	Surjectivity of I_0^*	196
	8.3	Stability Estimates Based on the Normal Operator	201
	8.4	The Normal Operator with a Matrix Weight	203
9	Inversion Formulas and Range		208
	9.1	Motivation	208
	9.2	Properties of Solutions of the Jacobi Equation	211
	9.3	The Smoothing Operator W	213
	9.4	Fredholm Inversion Formulas	218
	9.5	Revisiting the Euclidean Case	224
	9.6	Range	227
	9.7	Numerical Implementation	220

Contents ix

10	Tensor Tomography		233
	10.1	Holomorphic Integrating Factors	233
	10.2	Tensor Tomography	236
	10.3	Range for Tensors	238
11	Boundary Rigidity		241
	11.1	The Boundary Rigidity Problem	241
	11.2	Boundary Determination	244
	11.3	Determining the Lens Data and Volume	253
	11.4	Rigidity in a Given Conformal Class	256
	11.5	Determining the Dirichlet-to-Neumann Map	257
	11.6	Calderón Problem	260
	11.7	Boundary Rigidity for Simple Surfaces	267
12	The Attenuated Geodesic X-ray Transform		269
	12.1	The Attenuated X-ray Transform in the Plane	269
	12.2	Injectivity Results for Scalar Attenuations	271
	12.3	Surjectivity of I_{\perp}^*	275
	12.4	Discussion on General Weights	276
13	Non-Abelian X-ray Transforms		277
	13.1	Scattering Data	277
	13.2	Pseudo-linearization Identity	280
	13.3	Elementary Background on Connections	281
	13.4	Structure Equations Including a Connection	283
	13.5	Scattering Rigidity and Injectivity for Connections	286
	13.6	An Alternative Proof of Tensor Tomography	290
	13.7	General Skew-Hermitian Attenuations	293
	13.8	Injectivity for Connections and Higgs Fields	295
	13.9	Scattering Rigidity for Connections and Higgs Fields	298
	13.10	Matrix Holomorphic Integrating Factors	299
	13.11	Stability Estimate	302
14	Non-Abelian X-ray Transforms II		304
	14.1	Scattering Rigidity and Injectivity Results for $\mathfrak{gl}(n,\mathbb{C})$	304
	14.2	A Factorization Theorem from Loop Groups	308
	14.3	Proof of Theorems 14.1.1 and 14.1.2	310
	14.4	General Lie Groups	314
	14.5	Range of $I_{A,0}$ and $I_{A,\perp}$	317
	14.6	Surjectivity of $I_{A,0}^*$ and $I_{A,\perp}^*$	321
	14.7	Adding a Matrix Field	324

x Contents

15	Open Problems and Related Topics		326
	15.1	Open Problems	326
		Related Topics	327
	Refere	ences	332
	Index		342