

GRADUATE STUDIES
IN MATHEMATICS **250**

The Practice of Algebraic Curves

A Second Course in
Algebraic Geometry

David Eisenbud
Joe Harris



AMERICAN
MATHEMATICAL
SOCIETY

Providence, Rhode Island

Contents

Preface	xv
Introduction	1
Why you want to read this book	1
Why we wrote this book	2
What's with the title?	3
What's in this book	4
▸ Exercises and hints	7
▸ Relation of this book to other texts	7
Prerequisites, notation and conventions	8
▸ Commutative algebra	8
▸ Projective geometry	8
▸ Sheaves and cohomology	9
Chapter 1. Linear series and morphisms to projective space	11
1.1 Divisors	12
1.2 Divisors and rational functions	13
▸ Generalizations	13
▸ Divisors of functions	14
▸ Invertible sheaves	15
▸ Invertible sheaves and line bundles	17
1.3 Linear series and maps to projective space	18
1.4 The geometry of linear series	20
▸ An upper bound on $h^0(\mathcal{L})$	20
▸ Incomplete linear series	21
▸ Sums of linear series	23

• Which linear series define embeddings?	23
Exercises	26
Chapter 2. The Riemann–Roch theorem	29
2.1 How many sections?	29
• Riemann–Roch without duality	30
2.2 The most interesting linear series	31
• The adjunction formula	32
• Hurwitz’s theorem	34
2.3 Riemann–Roch with duality	37
• Residues	40
• Arithmetic genus and geometric genus	41
2.4 The canonical morphism	43
• Geometric Riemann–Roch	45
• Linear series on a hyperelliptic curve	46
2.5 Clifford’s theorem	47
2.6 Curves on surfaces	48
• The intersection pairing	48
• The Riemann–Roch theorem for smooth surfaces	49
• Blowups of smooth surfaces	50
2.7 Quadrics in $\mathbb{P}^3$ and the curves they contain	51
• The classification of quadrics	51
• Some classes of curves on quadrics	51
2.8 Exercises	52
Chapter 3. Curves of genus 0	57
3.1 Rational normal curves	58
3.2 Other rational curves	63
• Smooth rational quartics	64
• Some open problems about rational curves	66
3.3 The Cohen–Macaulay property	68
3.4 Exercises	71
Chapter 4. Smooth plane curves and curves of genus 1	75
4.1 Riemann, Clebsch, Brill and Noether	75
4.2 Smooth plane curves	77
• 4.2.1 Differentials on a smooth plane curve	77
• 4.2.2 Linear series on a smooth plane curve	79
• 4.2.3 The Cayley–Bacharach–Macaulay theorem	80
4.3 Curves of genus 1 and the group law of an elliptic curve	82

4.4	Low degree divisors on curves of genus 1	84
▸	The dimension of families	84
▸	Double covers of $\mathbb{P}^1$	85
▸	Plane cubics	85
4.5	Genus 1 quartics in $\mathbb{P}^3$	86
4.6	Genus 1 quintics in $\mathbb{P}^4$	88
4.7	Exercises	90
Chapter 5. Jacobians		93
5.1	Symmetric products and the universal divisor	94
▸	Finite group quotients	95
5.2	The Picard varieties	96
5.3	Jacobians	98
5.4	Abel's theorem	101
5.5	The $g + 3$ theorem	103
5.6	The schemes $W_d^r(C)$	105
5.7	Examples in low genus	105
▸	Genus 1	105
▸	Genus 2	106
▸	Genus 3	106
5.8	Martens' theorem	106
5.9	Exercises	108
Chapter 6. Hyperelliptic curves and curves of genus 2 and 3		111
6.1	Hyperelliptic curves	111
▸	The equation of a hyperelliptic curve	111
▸	Differentials on a hyperelliptic curve	113
6.2	Branched covers with specified branching	114
▸	Branched covers of $\mathbb{P}^1$	115
6.3	Curves of genus 2	117
▸	Maps of C to $\mathbb{P}^1$	117
▸	Maps of C to $\mathbb{P}^2$	118
▸	Embeddings in $\mathbb{P}^3$	119
▸	The dimension of the family of genus 2 curves	120
6.4	Curves of genus 3	121
▸	Other representations of a curve of genus 3	121
6.5	Theta characteristics	123
▸	Counting theta characteristics (proof of Theorem 6.8)	127
6.6	Exercises	129

Chapter 7. Fine moduli spaces	133
7.1 What is a moduli problem?	133
7.2 What is a solution to a moduli problem?	136
7.3 Hilbert schemes	137
▸ 7.3.1 The tangent space to the Hilbert scheme	138
▸ 7.3.2 Parametrizing twisted cubics	140
▸ 7.3.3 Construction of the Hilbert scheme in general	141
▸ 7.3.4 Grassmannians	142
▸ 7.3.5 Equations defining the Hilbert scheme	143
7.4 Bounding the number of maps between curves	144
7.5 Exercises	146
Chapter 8. Moduli of curves	147
8.1 Curves of genus 1	147
▸ M_1 is a coarse moduli space	148
▸ The good news	149
▸ Compactifying M_1	150
8.2 Higher genus	152
▸ Stable, semistable, unstable	154
8.3 Stable curves	155
▸ How we deal with the fact that $\overline{M}_g$ is not fine	157
8.4 Can one write down a general curve of genus g ?	157
8.5 Hurwitz spaces	159
▸ The dimension of M_g	160
▸ Irreducibility of M_g	161
8.6 The Severi variety	161
▸ Local geometry of the Severi variety	162
8.7 Exercises	165
Chapter 9. Curves of genus 4 and 5	167
9.1 Curves of genus 4	167
▸ The canonical model	167
▸ Maps to projective space	168
9.2 Curves of genus 5	172
9.3 Canonical curves of genus 5	173
▸ First case: the intersection of the quadrics has dimension 1	173
▸ Second case: the intersection of the quadrics is a surface	176
9.4 Exercises	177
Chapter 10. Hyperplane sections of a curve	179

10.1	Linearly general position	179
10.2	Castelnuovo's theorem	183
	▸ Proof of Castelnuovo's bound	184
	▸ Consequences and special cases	188
10.3	Other applications of linearly general position	189
	▸ Existence of good projections	189
	▸ The case of equality in Martens' theorem	190
	▸ The $g + 2$ theorem	192
10.4	Exercises	194
Chapter 11.	Monodromy of hyperplane sections	197
11.1	Uniform position and monodromy	197
	▸ The monodromy group of a generically finite morphism	198
	▸ Uniform position	199
11.2	Flexes and bitangents are isolated	200
	▸ Not every tangent line is tangent at a flex	200
	▸ Not every tangent is bitangent	201
11.3	Proof of the uniform position lemma	201
	▸ Uniform position for higher-dimensional varieties	203
11.4	Applications of uniform position	204
	▸ Irreducibility of fiber powers	204
	▸ Numerical uniform position	204
	▸ Sums of linear series	205
	▸ Nodes of plane curves	205
11.5	Exercises	206
Chapter 12.	Brill–Noether theory and applications to genus 6	209
12.1	What linear series exist?	209
12.2	Brill–Noether theory	209
	▸ 12.2.1 A Brill–Noether inequality	211
	▸ 12.2.2 Refinements of the Brill–Noether theorem	212
12.3	Linear series on curves of genus 6	215
	▸ 12.3.1 General curves of genus 6	216
	▸ 12.3.2 Del Pezzo surfaces	217
	▸ 12.3.3 The canonical image of a general curve of genus 6	219
12.4	Classification of curves of genus 6	219
	▸ $ D $ has a basepoint	220
	▸ C is not trigonal and the image of ϕ_D is two-to-one onto a plane curve of degree 3.	220
12.5	Exercises	221

Chapter 13. Inflection points	223
13.1 Inflection points, Plücker formulas and Weierstrass points	223
• Definitions	223
• The Plücker formula	224
• Flexes of plane curves	226
• Weierstrass points	226
• Another characterization of Weierstrass points	227
13.2 Finiteness of the automorphism group	228
13.3 Curves with automorphisms are special	230
13.4 Inflections of linear series on $\mathbb{P}^1$	231
• Schubert cycles	232
• Special Schubert cycles and Pieri's formula	234
• Conclusion	235
13.5 Exercises	237
Chapter 14. Proof of the Brill–Noether theorem	241
14.1 Castelnuovo's approach	241
• Upper bound on the codimension of $W_d^r(C)$	243
14.2 Specializing to a g -cuspidal curve	244
• Constructing curves with cusps	244
• Smoothing a cuspidal curve	244
14.3 The family of Picard varieties	245
• The Picard variety of a cuspidal curve	245
• The relative Picard variety	246
• Limits of invertible sheaves	247
14.4 Putting it all together	250
• Nonexistence	250
• Existence	250
14.5 Brill–Noether with inflection	250
14.6 Exercises	252
Chapter 15. Using a singular plane model	253
15.1 Nodal plane curves	253
• 15.1.1 Differentials on a nodal plane curve	254
• 15.1.2 Linear series on a nodal plane curve	256
15.2 Arbitrary plane curves	259
• The conductor ideal and linear series on the normalization	260
• Differentials	262
15.3 Exercises	265
Chapter 16. Linkage and the canonical sheaves of singular curves	269

16.1	Introduction	269
16.2	Linkage of twisted cubics	270
16.3	Linkage of smooth curves in $\mathbb{P}^3$	272
16.4	Linkage of purely 1-dimensional schemes in $\mathbb{P}^3$	273
16.5	Degree and genus of linked curves	274
	▸ Dualizing sheaves for singular curves	274
16.6	The construction of dualizing sheaves	275
16.7	The linkage equivalence relation	280
16.8	Comparing the canonical sheaf with that of the normalization	281
16.9	A general Riemann–Roch theorem	284
16.10	Exercises	285
	▸ Ropes and ribbons	286
	▸ General adjunction	288
Chapter 17.	Scrolls and the curves they contain	289
17.1	Some classical geometry	290
17.2	1-generic matrices and the equations of scrolls	292
17.3	Scrolls as images of projective bundles	298
17.4	Curves on a 2-dimensional scroll	299
	▸ Finding a scroll containing a given curve	299
	▸ Finding curves on a given scroll	301
17.5	Exercises	306
Chapter 18.	Free resolutions and canonical curves	309
18.1	Free resolutions	309
	▸ The classification of 1-generic $2 \times f$ matrices	311
	▸ How to look at a resolution	313
	▸ When is a finite free complex a resolution?	313
18.2	Depth and the Cohen–Macaulay property	315
	▸ The Gorenstein property	316
18.3	The Eagon–Northcott complex	317
	▸ The case $\text{rank } G = 1$: the Koszul complex	318
	▸ The case $\text{rank } F = \text{rank } G + 1$: the Hilbert–Burch complex	320
	▸ The Hilbert–Burch theorem	321
	▸ The general case of the Eagon–Northcott complex	322
18.4	Green’s conjecture	326
18.5	Exercises	331
Chapter 19.	Hilbert Schemes	335

19.1	Degree 3	335
▸	The other component of $\mathcal{H}_{0,3,3}$	336
19.2	Extraneous components	337
19.3	Degree 4	338
▸	Genus 0	338
▸	Genus 1	339
19.4	Degree 5	339
▸	Genus 2	339
19.5	Degree 6	340
▸	Genus 4	341
▸	Genus 3	341
19.6	Degree 7	341
19.7	The expected dimension of $\mathcal{H}_{g,r,d}^\circ$	341
19.8	Some open problems	343
▸	Brill–Noether in low codimension	343
▸	Maximally special curves	344
▸	Rigid curves?	345
19.9	Degree 8, genus 9	346
19.10	Degree 9, genus 10	347
19.11	Estimating the dimension of the restricted Hilbert schemes using the Brill–Noether theorem	348
19.12	Exercises	349
Appendix: A historical essay on some topics in algebraic geometry by Jeremy Gray		353
A.1	Greek mathematicians and conic sections	353
A.2	The first appearance of complex numbers	355
A.3	Conic sections from the 17th to the 19th centuries	356
A.4	Curves of higher degree from the 17th to the early 19th century	359
A.5	The birth of projective space	367
A.6	Riemann’s theory of algebraic curves and its reception	368
A.7	First ideas about the resolution of singular points	370
A.8	The work of Brill and Noether	372
A.9	Historical bibliography	373
Hints to marked exercises		379
Bibliography		391
Index		401