

# K-stability of Fano Varieties

CHENYANG XU  
*Princeton University*



---

# Contents

<i>Preface</i>	<i>page</i> vii
<i>Notation and Conventions</i>	ix
<b>Preliminaries</b>	1
<b>1 Higher Dimensional Geometry Background</b>	12
1.1 Okounkov Body	12
1.2 Valuations	26
1.3 Asymptotic Invariants	37
1.4 Minimal Model Program and Boundedness	45
Exercises	51
<b>2 K-stability via Test Configurations</b>	54
2.1 Test Configuration and Invariants	54
2.2 T-variety and Product Test Configurations	68
2.3 Special Test Configurations	75
Exercises	89
<b>3 K-stability via Filtrations</b>	92
3.1 Filtered Linear Series	92
3.2 S-invariants on Filtrations	101
3.3 Log Canonical Slopes	108
3.4 Approximation of Filtrations	119
3.5 * Relative Study of Two Filtrations	127
Exercises	134
<b>4 K-stability via Valuations</b>	138
4.1 Fujita–Li’s Valuative Criterion	138
4.2 Geometry of Special Valuations	149
4.3 Minimizer of $\delta(X, \Delta)$	156
4.4 * Equivariant Stability	172

4.5	* Abban–Zhuang Method	180
	Exercises	202
<b>5</b>	<b>Higher Rank Finite Generation</b>	207
	5.1 Multistep Degenerations	207
	5.2 Finite Generation for Quasi-monomial Valuations	218
	Exercises	235
<b>6</b>	<b>Reduced Stability</b>	238
	6.1 Twisting Filtrations and Valuations	238
	6.2 Reduced Uniform Stability	249
	6.3 Stability Threshold $\delta_{\mathbb{T}}$	253
	Exercises	259
<b>7</b>	<b>K-moduli Stack</b>	260
	7.1 Family of K-stable Log Fano Pairs	260
	7.2 Boundedness of Log Fano Pairs	266
	7.3 Openness of K-semistability	267
	7.4 The K-moduli Stack	271
	7.5 * Twisted K-stability	274
	Exercises	285
<b>8</b>	<b>K-moduli Space</b>	288
	8.1 Good Moduli Space	288
	8.2 K-moduli Space $X_{n,N,V}^{\mathbb{K}}$	295
	8.3 Properness of K-moduli	305
	Exercises	319
<b>9</b>	<b>Positivity of the CM Line Bundle</b>	323
	9.1 Harder–Narasimhan Filtration for a Family	323
	9.2 Semi-positivity of CM Line Bundles	335
	9.3 Twisted Families	343
	9.4 Positivity of CM Line Bundle	348
	Exercises	362
	<i>Appendix A Solutions to Exercises</i>	365
	<i>Glossary</i>	393
	<i>Bibliography</i>	399
	<i>Index</i>	409