

Proven Impossible

Elementary Proofs of Profound Impossibility from
Arrow, Bell, Chaitin, Gödel, Turing and More

DAN GUSFIELD

University of California, Davis



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