

STABLE AND CONSERVATIVE NUMERICAL SCHEMES FOR HYPERBOLIC PROBLEMS USING NON-CONVEX POLYGONAL MESHES

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde (Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

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Zürich, 2022

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