

Mathematical Logic and Computation

Jeremy Avigad
Carnegie Mellon University



Contents

<i>Preface</i>	page ix
<i>Acknowledgments</i>	xii
1 Fundamentals	1
1.1 Languages and Structures	2
1.2 Inductively Defined Sets	4
1.3 Terms and Formulas	8
1.4 Trees	11
1.5 Structural Recursion	15
1.6 Bound Variables	21
2 Propositional Logic	28
2.1 The Language of Propositional Logic	28
2.2 Axiomatic Systems	30
2.3 The Provability Relation	34
2.4 Natural Deduction	36
2.5 Some Propositional Validities	42
2.6 Normal Forms for Classical Logic	47
2.7 Translations Between Logics	50
2.8 Other Deductive Systems	53
3 Semantics of Propositional Logic	59
3.1 Classical Logic	60
3.2 Algebraic Semantics for Classical Logic	65
3.3 Intuitionistic Logic	70
3.4 Algebraic Semantics for Intuitionistic Logic	75
3.5 Variations	79
4 First-Order Logic	83
4.1 The Language of First-Order Logic	83
4.2 Quantifiers	85
4.3 Equality	90
4.4 Equational and Quantifier-Free Logic	94
4.5 Normal Forms for Classical Logic	96

4.6	Translations Between Logics	97
4.7	Definite Descriptions	101
4.8	Sorts and Undefined Terms	105
5	Semantics of First-Order Logic	110
5.1	Classical Logic	110
5.2	Equational and Quantifier-Free Logic	117
5.3	Intuitionistic Logic	118
5.4	Algebraic Semantics	123
5.5	Definability	127
5.6	Some Model Theory	132
6	Cut Elimination	138
6.1	An Intuitionistic Sequent Calculus	138
6.2	Classical Sequent Calculi	143
6.3	Cut-Free Completeness of Classical Logic	147
6.4	Cut Elimination for Classical Logic	151
6.5	Cut Elimination for Intuitionistic Logic	156
6.6	Equality	158
6.7	Variations on Cut Elimination	161
6.8	Cut-Free Completeness of Intuitionistic Logic	162
7	Properties of First-Order Logic	166
7.1	Herbrand's Theorem	166
7.2	Explicit Definability and the Disjunction Property	172
7.3	Interpolation Theorems	174
7.4	Indefinite Descriptions	180
7.5	Skolemization in Classical Theories	186
8	Primitive Recursion	192
8.1	The Primitive Recursive Functions	192
8.2	Some Primitive Recursive Functions and Relations	195
8.3	Finite Sets and Sequences	199
8.4	Other Recursion Principles	204
8.5	Recursion along Well-Founded Relations	208
8.6	Diagonalization and Reflection	210
9	Primitive Recursive Arithmetic	214
9.1	A Quantifier-Free Axiomatization	214
9.2	Bootstrapping PRA	218
9.3	Finite Sets and Sequences	224
9.4	First-Order PRA	229
9.5	Equational PRA	231

10	First-Order Arithmetic	237
10.1	Peano Arithmetic and Heyting Arithmetic	237
10.2	The Arithmetic Hierarchy	240
10.3	Subsystems of First-Order Arithmetic	247
10.4	Interpreting PRA	252
10.5	Truth and Reflection	258
10.6	Conservation Results via Cut Elimination	266
10.7	Conservation Results via Model Theory	272
11	Computability	279
11.1	The Computable Functions	279
11.2	Computability and Arithmetic Definability	281
11.3	Undecidability and the Halting Problem	284
11.4	Computably Enumerable Sets	290
11.5	The Recursion Theorem	293
11.6	Turing Machines	296
11.7	The Lambda Calculus	301
11.8	Relativized Computation	308
11.9	Computability and Infinite Binary Trees	314
12	Undecidability and Incompleteness	322
12.1	Computability and Representability	323
12.2	Incompleteness via Undecidability	329
12.3	Incompleteness via Self-Reference	334
12.4	The Second Incompleteness Theorem	336
12.5	Some Decidable Theories	341
13	Finite Types	351
13.1	The Simply Typed Lambda Calculus	351
13.2	Strong Normalization	356
13.3	Confluence	362
13.4	Combinatory Logic	364
13.5	Equational Theories	368
13.6	First-Order Theories and Models	371
13.7	Primitive Recursive Functionals	376
13.8	Propositions as Types	379
14	Arithmetic and Computation	383
14.1	Realizability	383
14.2	Metamathematical Applications	389
14.3	Modified Realizability	394
14.4	Finite-Type Arithmetic	399
14.5	The Dialectica Interpretation	401

15	Second-Order Logic and Arithmetic	409
15.1	Second-Order Logic	410
15.2	Semantics of Second-Order Logic	413
15.3	Cut Elimination	417
15.4	Second-Order Arithmetic	422
15.5	The Analytic Hierarchy	426
15.6	The Second-Order Typed Lambda Calculus	430
15.7	Higher-Order Logic and Arithmetic	437
16	Subsystems of Second-Order Arithmetic	440
16.1	Arithmetic Comprehension	441
16.2	Recursive Comprehension	446
16.3	Formalizing Analysis	449
16.4	Weak König's Lemma	457
16.5	Π_1^1 Comprehension and Inductive Definitions	461
16.6	Arithmetic Transfinite Recursion	464
17	Foundations	472
17.1	Simple Type Theory	472
17.2	Mathematics in Simple Type Theory	475
17.3	Set Theory	475
17.4	Mathematics in Set Theory	477
17.5	Dependent Type Theory	478
17.6	Inductive Types	481
17.7	Mathematics in Dependent Type Theory	485
Appendix	Background	488
A.1	Naive Set Theory	488
A.2	Orders and Equivalence Relations	490
A.3	Cardinality and Zorn's Lemma	491
A.4	Topology	493
A.5	Category Theory	495
	<i>References</i>	497
	<i>Notation</i>	504
	<i>Index</i>	508