## Homological Methods in Banach Space Theory

FÉLIX CABELLO SÁNCHEZ

Universidad de Extremadura

JESÚS M. F. CASTILLO
Universidad de Extremadura



## Contents

	Preja	Prejace		
	Preli	Preliminaries		
1	Complemented Subspaces of Banach Spaces			
	1.1	Banach and Quasi-Banach Spaces		
	1.2	Complemented Subspaces		
	1.3	Uncomplemented Subspaces		
	1.4	Local Properties and Techniques		
	1.5	The Dunford-Pettis, Grothendieck, Pełczyński		
		and Rosenthal Properties		
	1.6	C(K)-Spaces and Their Complemented Subspaces		
	1.7	Sobczyk's Theorem and Its Derivatives		
	1.8	Notes and Remarks	36	
		1.8.1 Topological Stuff	36	
		1.8.2 Orlicz, Young, Fenchel and L <sub>0</sub> Too	38	
		1.8.3 Ultrapowers of $L_p$ When $0$	39	
		1.8.4 Sobczyk's Theorem Strikes Back	42	
2	The l	e Language of Homology		
	2.1	Exact Sequences of Quasi-Banach Spaces		
	2.2	Basic Examples of Exact Sequences		
	2.3	Topologically Exact Sequences		
	2.4	Categorical Constructions for Absolute Beginners		
	2.5	Pullback and Pushout		
	2.6	Pushout and Exact Sequences		
	2.7	Projective Presentations: the Universal Property of $\ell_p$		
	2.8	Pullbacks and Exact Sequences		
	2.9	Injective Presentations: the Universal Property of $\ell_{\infty}$ 8		
	2.10	All about That Pullback/Pushout Diagram		

vi Contents

	2.11	Diagonal and Parallel Principles		
	2.12	Homological Constructions Appearing in Nature		
	2.13	The Device		
	2.14	Extension and Lifting of Operators		
	2.15	Notes and Remarks	119	
		2.15.1 Categorical Limits	119	
		2.15.2 How to Draw More Diagrams	120	
		2.15.3 Amalgamation of Sequences	124	
		2.15.4 Categories of Short Exact Sequences	125	
3	Quasilinear Maps			
	3.1	An Introduction to Quasilinear Maps		
	3.2	Quasilinear Maps in Action		
	3.3	Quasilinear Maps versus Exact Sequences		
	3.4	Local Convexity of Twisted Sums and X-Spaces		
	3.5	The Pullback and Pushout in Quasilinear Terms		
	3.6	Spaces of Quasilinear Maps		
	3.7	Homological Properties of $\ell_p$ and $L_p$ When $0$		
	3.8	Exact Sequences of Banach Spaces and Duality		
	3.9	Different Versions of a Quasilinear Map		
	3.10	Linearisation of Quasilinear Maps		
	3.11	The Type of Twisted Sums		
	3.12	A Glimpse of Centralizers		
	3.13	Notes and Remarks		
		3.13.1 Domański's Work on Quasilinear Maps	190	
		3.13.2 A Cohomological Approach to Quasilinearity	193	
		3.13.3 Table of Correspondences between Diagrams		
		and Quasilinear Maps	194	
4	The Functor Ext and the Homology Sequences			
	4.1	The Functor Ext		
	4.2	The Homology Sequences		
	4.3	Homology in Quasilinear Terms		
	4.4	Alternative Constructions of Ext		
	4.5	Topological Aspects of Ext		
	4.6	Notes and Remarks	234	
		4.6.1 Adjoint Functors	234	
		4.6.2 Derived Functors	237	
		4.6.3 Unknown Knowns about Ext <sup>2</sup>	240	
		4.6.4 Open Problems about the Topology of Ext	241	

Contents	V11

5	Local Methods in the Theory of Twisted Sums			243	
	5.1	Local S	Splitting	244	
	5.2	Uniform	m Boundedness Principles for Exact Sequences	258	
	5.3	The Mysterious Role of the BAP			
	5.4	Notes a	and Remarks	283	
		5.4.1	Which Banach Spaces Are <i>K</i> -Spaces?	283	
		5.4.2	Twisting a Few Exotic Banach Spaces	284	
6	Fraïssé Limits by the Pound				
	6.1	Fraïssé	Classes and Fraïssé Sequences	288	
7	6.2	Almost Universal Disposition			
	6.3	Almost Universal Complemented Disposition			
	6.4	A Universal Operator on G <sub>p</sub>			
	6.5	Notes a	324		
		6.5.1	What If $\varepsilon = 0$ ?	324	
		6.5.2	Before $G_p$ Spaces Fade Out	325	
		6.5.3	Fraïssé Classes of Banach Spaces	326	
7	Extension of Operators, Isomorphisms and Isometries				
	7.1	Operators: Extensible and UFO Spaces			
	7.2	Isomor	phisms: the Automorphic Space Problem	336	
	7.3	Isometries: Universal Disposition			
	7.4	Positio	ns in Banach Spaces	354	
	7.5	Notes and Remarks			
		7.5.1	Isomorphic but Different Twisted Sums	365	
		7.5.2	How Many Twisted Sums of Two Spaces Exist?	366	
		7.5.3	Moving towards the Automorphic Space Problem	368	
		7.5.4	The Product of Spaces of (Almost) Universal		
			Disposition	369	
8	Extension of $C(K)$ -Valued Operators				
	8.1	Zippin Selectors			
	8.2	The Lin	ndenstrauss-Pełczyński Theorem	378	
	8.3	Kalton	's Approach to the $\mathscr{C}$ -Extension Property	383	
	8.4	Sequen	ce Spaces with the &-Extension Property	3 <b>9</b> 4	
	8.5	€-Extensible Spaces			
	8.6	The Dark Side of the Johnson-Zippin Theorem			
	8.7	The Astounding Story behind the CCKY Problem			
	8.8	3.8 Notes and Remarks		434	
		8.8.1	Homogeneous Zippin Selectors	434	
		8.8.2	Lindenstrauss-Valued Extension Results	436	

viii Contents

		8.8.3	The Last Stroke on the Extension of &-Valued		
			Lipschitz Maps	437	
		8.8.4	Property (M) and M-Ideals	440	
		8.8.5	Set Theoretic Axioms and Twisted Sum Affairs	440	
9	Singular Exact Sequences				
	9.1	Basic Properties and Techniques			
	9.2	Singular Quasilinear Maps			
	9.3	Amalgamation Techniques			
	9.4	Notes	and Remarks	462	
		9.4.1	Super Singularity	462	
		9.4.2	Disjoint Singularity	463	
		9.4.3	Cosingularity	465	
		9.4.4	The Basic Sequence Problem	465	
10	Back to Banach Space Theory			468	
	10.1	Vector-Valued Versions of Sobczyk's Theorem			
	10.2	Polyhedral $\mathscr{L}_{\infty}$ -Spaces			
	10.3	Lipschitz and Uniformly Homeomorphic $\mathscr{L}_{\infty}$ -Spaces			
	10.4				
	10.5	3-Space Problems			
	10.6	Extension of $\mathscr{L}_{\infty}$ -Valued Operators			
	10.7	Kadec Spaces			
	10.8	The Kalton-Peck Spaces			
	10.9	The Properties of $Z_2$ Explained by Itself			
	Bibliography			521	
	Index			543	