An Introduction to Optimization on Smooth Manifolds

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Preface

Optimization problems on smooth manifolds arise in science and engineering as a result of natural geometry (e.g., the set of orientations of physical objects in space is a manifold), latent data simplicity (e.g., high-dimensional data points lie close to a low-dimensional linear subspace, leading to low-rank data matrices), symmetry (e.g., observations are invariant under rotation, translation or other group actions, leading to quotients) and positivity (e.g., covariance matrices and diffusion tensors are positive definite). This has led to successful applications notably in machine learning, computer vision, robotics, scientific computing, dynamical systems and signal processing.

Accordingly, optimization on manifolds has garnered increasing interest from researchers and engineers alike. Building on 50 years of research efforts that have recently intensified, it is now recognized as a wide, beautiful and effective generalization of unconstrained optimization on linear spaces.

Yet, engineering programs seldom include training in differential geometry, that is, the field of mathematics concerned with smooth manifolds. Moreover, existing textbooks on this topic usually align with the interests of mathematicians more than with the needs of engineers and applied mathematicians. This creates a significant but avoidable barrier to entry for optimizers.

One of my goals in writing this book is to offer a different, if at times unorthodox, introduction to differential geometry. Definitions and tools are introduced in a need-based order for optimization. We start with a restricted setting—that of embedded submanifolds of linear spaces—which allows us to define all necessary concepts in direct reference to their usual counterparts from linear spaces. This covers a wealth of applications.

In what is perhaps the clearest departure from standard exposition, charts and atlases are not introduced until quite late. The reason for doing so is twofold: pedagogically, charts and atlases are more abstract than what is needed to work on embedded submanifolds; and pragmatically, charts are seldom if ever useful in practice. It would be unfortunate to give them center stage.

Of course, charts and atlases are the right tool to provide a unified treatment of all smooth manifolds in an intrinsic way. They are introduced eventually, at which point it becomes possible to discuss quotient manifolds: a powerful language to understand symmetry in optimization. Perhaps this abstraction is necessary to fully appreciate the depth of optimization on manifolds as more than just a fancy tool for