

William M. Farmer

Simple Type Theory

A Practical Logic for Expressing
and Reasoning About Mathematical Ideas

 Birkhäuser

Contents

Preface	xiii
1 Introduction	1
Summary of the Contents	3
2 Answers to Readers' Questions	6
2.1 Why Logic?	6
2.2 Why a Practical Logic?	6
2.3 Why Simple Type Theory?	7
2.4 Why not First-Order Logic?	8
2.5 Why not Set Theory?	9
2.6 Why not Dependent Type Theory?	10
2.7 Why Undefinedness?	11
2.8 Why Model Theory instead of Proof Theory?	11
3 Preliminary Concepts	13
3.1 What is Mathematics?	13
3.2 Mathematical Values	15
3.2.1 Sets	16
3.2.2 Sequences	17
3.2.3 Relations	18
3.2.4 Functions	19
3.2.5 Boolean Values and Predicates	21
3.3 Binders	22
3.4 Undefinedness	25
3.5 Mathematical Structures	27
3.6 Examples of Mathematical Structures	30
3.7 Conclusions	32
3.8 Exercises	32

4	Syntax	36
4.1	Notation	36
4.2	Symbols	37
4.3	Types	38
4.4	Expressions	40
4.5	Bound and Free Variables	42
4.6	Substitution	42
4.7	Languages	44
4.8	Conclusions	44
4.9	Exercises	45
5	Semantics	47
5.1	Interpretations	47
5.2	General Models	48
5.3	Finite General Models	52
5.4	Standard Models	53
5.5	Satisfiability, Validity, and Semantic Consequence	54
5.6	Isomorphic General Models	56
5.7	Expansion of a General Model	57
5.8	Standard vs. General Semantics	58
5.9	Examples of Standard Models	59
5.10	Conclusions	62
5.11	Exercises	62
6	Additional Notation	64
6.1	Boolean Operators	64
6.2	Binary Operators	67
6.3	Quantifiers	68
6.4	Definedness	69
6.5	Sets	70
6.6	Tuples	70
6.7	Functions	70
6.8	Miscellaneous Notation	70
6.9	Quasitypes and Dependent Quasitypes	72
6.10	Conclusions	75
6.11	Exercises	75
7	Beta-Reduction and Substitution	77
7.1	Beta-Reduction	77
7.2	Universal Instantiation	80

7.3	Invalid Beta-Reduction	81
7.4	Alpha-Conversion	82
7.5	Conclusions	83
7.6	Exercises	83
8	Proof Systems	84
8.1	Background	84
8.2	A Proof System for Alonzo	86
8.2.1	Axioms	86
8.2.2	Rules of Inference	88
8.2.3	Proofs	88
8.3	Soundness	89
8.4	Frugal General Models	89
8.5	Completeness	90
8.6	Conclusions	93
8.7	Exercises	93
9	Theories	94
9.1	Axiomatic Theories	94
9.2	Theory Extensions	101
9.3	Conservative Theory Extensions	104
9.4	Categorical Theories	111
9.5	Complete Theories	115
9.6	Fundamental Form of a Mathematical Problem	120
9.7	Model Theory	121
9.8	Conclusions	122
9.9	Exercises	124
10	Sequences	127
10.1	Systems of Natural Numbers	127
10.2	Notation for Sequences	128
10.3	Conclusions	130
10.4	Exercises	130
11	Developments	131
11.1	Theory Developments	131
11.2	Development of Natural Number Arithmetic	134
11.2.1	Basic Definitions and Theorems	134
11.2.2	Commutative Semiring	135
11.2.3	Weak Total Order	136

11.2.4	Divides Lattice	136
11.3	Conclusions	138
11.4	Exercises	138
12	Real Number Mathematics	139
12.1	Complete Ordered Fields	140
12.2	Alternatives to the Construction of COF	143
12.3	Development of Real Number Mathematics	145
12.3.1	Some Basic Definitions and Theorems	145
12.3.2	Naturals, Integers, and Rationals	146
12.3.3	Iterated Sum and Product Operators	147
12.3.4	Calculus	148
12.3.5	Euclidean Space	150
12.4	Skolem's Paradox	153
12.5	Conclusions	154
12.6	Exercises	154
13	Morphisms	156
13.1	A Motivating Example	156
13.2	The Little Theories Method	158
13.3	Theory Morphisms	159
13.3.1	Theory Translations	159
13.3.2	Morphism Theorem	164
13.3.3	Examples of Theory Morphisms	167
13.3.4	Faithful Theory Morphisms	175
13.4	Development Morphisms	177
13.4.1	Development Translations	178
13.4.2	Transportations	182
13.5	Mathematics Libraries	187
13.5.1	Theory Graphs	187
13.5.2	Development Graphs	188
13.5.3	Realm Graphs	188
13.6	Theory Graph Combinators	189
13.7	Conclusions	194
13.8	Exercises	195
14	Alonzo Variants	196
14.1	Alonzo with Indefinite Description	196
14.1.1	Introduction	196
14.1.2	Syntax	197

14.1.3	Semantics	197
14.1.4	Proof System	197
14.1.5	Theorems	198
14.2	Alonzo with Sorts	199
14.2.1	Introduction	199
14.2.2	Syntax	199
14.2.3	Semantics	201
14.2.4	Proof System	203
14.2.5	Theorems	204
14.3	Alonzo with Quotation and Evaluation	204
14.3.1	Introduction	204
14.3.2	Syntax	205
14.3.3	Semantics	207
14.3.4	Proof System	208
14.3.5	Theorems	208
14.4	Conclusions	209
15	Software Support	210
15.1	Basis Support	211
15.2	Advanced Support	212
15.2.1	Organization	212
15.2.2	Inference	213
15.2.3	Computation	213
15.2.4	Concretization	214
15.2.5	Narration	214
15.3	Fully Integrated Support	215
15.4	Conclusions	215
A	Metatheorems of $\mathfrak{A}$	216
A.1	Universal Instantiation	216
A.2	Alpha-Conversion	221
A.3	Substitution Rule	223
A.4	Tautology Theorems	228
A.5	Deduction Theorem	234
A.6	Miscellaneous Metatheorems	238
B	Soundness of $\mathfrak{A}$	240
C	Henkin's Theorem for $\mathfrak{A}$	250

Bibliography	262
List of Figures	274
List of Tables	275
List of Theorems, Examples, Remarks, and Modules	276
Index	285

List of Figures

3.1	The Tetrapod Model of Mathematical Knowledge	15
5.1	Kinds of General Models	54
8.1	Kinds of General Models of Languages of Power ω	90
9.1	Kinds of Theories	124
13.1	An Example Theory Graph	177
13.2	Input and Output Graph of thy-ext	191
13.3	Input and Output Graph of thy-ren	192
13.4	Input and Output Graph of thy-comb	193
13.5	Input and Output Graph of thy-inst	194

List of Tables

- 3.1 Truth Table for Boolean Operators 21
- 4.1 Notational Conventions 37
- 4.2 Notational Definitions for Types 39
- 4.3 Notational Definitions for Expressions 39
- 4.4 Kinds of Expressions 41
- 6.1 Notational Definitions for Boolean Operators 65
- 6.2 Notational Definitions for Binary Operators 68
- 6.3 Notational Definitions for Quantifiers 68
- 6.4 Notational Definitions for Definedness 69
- 6.5 Notational Definitions for Sets 71
- 6.6 Notational Definitions for Tuples 71
- 6.7 Notational Definitions for Functions 72
- 6.8 Miscellaneous Notational Definitions 73
- 6.9 Notational Definitions for Quasitypes 74
- 6.10 Notational Definitions for Dependent Quasitypes 75
- 9.1 Kinds of Mathematical Knowledge Modules 96
- 9.2 Kinds of Theory Extensions 102
- 9.3 Theories Presented in Chapter 9 123
- 10.1 Notational Definitions for Sequences 129
- 12.1 Notational Definitions for Real Numbers 140
- 14.1 Notational Definition for Indefinite Description 197
- 14.2 Notational Definitions for Sorts 200
- 14.3 Additional Notational Definitions for AlonzoS 203
- 14.4 Notational Definitions for AlonzoQE 206
- 14.5 Logical Constants of AlonzoQE 206