

Jean-François Le Gall

Measure Theory, Probability, and Stochastic Processes

Contents

Part I Measure Theory

1 Measurable Spaces	3
1.1 Measurable Sets	3
1.2 Positive Measures	6
1.3 Measurable Functions	10
1.4 Monotone Class	13
1.5 Exercises	15
2 Integration of Measurable Functions.....	17
2.1 Integration of Nonnegative Functions	17
2.2 Integrable Functions	27
2.3 Integrals Depending on a Parameter	31
2.4 Exercises	35
3 Construction of Measures	41
3.1 Outer Measures	41
3.2 Lebesgue Measure	44
3.3 Relation with Riemann Integrals	53
3.4 A Subset of \mathbb{R} Which Is Not Measurable	55
3.5 Finite Measures on \mathbb{R} and the Stieltjes Integral	56
3.6 The Riesz-Markov-Kakutani Representation Theorem	58
3.7 Exercises	59
4 L^p Spaces.....	63
4.1 Definitions and the Hölder Inequality	63
4.2 The Banach Space $L^p(E, \mathcal{A}, \mu)$	67
4.3 Density Theorems in L^p Spaces	71
4.4 The Radon-Nikodym Theorem	75
4.5 Exercises	81

5	Product Measures	85
5.1	Product σ -Fields	85
5.2	Product Measures	87
5.3	The Fubini Theorems	90
5.4	Applications	94
5.4.1	Integration by Parts	94
5.4.2	Convolution	95
5.4.3	The Volume of the Unit Ball	99
5.5	Exercises	101
6	Signed Measures	105
6.1	Definition and Total Variation	105
6.2	The Jordan Decomposition	109
6.3	The Duality Between L^p and L^q	113
6.4	The Riesz-Markov-Kakutani Representation Theorem for Signed Measures	118
6.5	Exercises	119
7	Change of Variables	121
7.1	The Change of Variables Formula	121
7.2	The Gamma Function	127
7.3	Lebesgue Measure on the Unit Sphere	128
7.4	Exercises	130

Part II Probability Theory

8	Foundations of Probability Theory	135
8.1	General Definitions	136
8.1.1	Probability Spaces	136
8.1.2	Random Variables	138
8.1.3	Mathematical Expectation	140
8.1.4	An Example: Bertrand's Paradox	144
8.1.5	Classical Laws	146
8.1.6	Distribution Function of a Real Random Variable	149
8.1.7	The σ -Field Generated by a Random Variable	150
8.2	Moments of Random Variables	151
8.2.1	Moments and Variance	151
8.2.2	Linear Regression	155
8.2.3	Characteristic Functions	156
8.2.4	Laplace Transform and Generating Functions	160
8.3	Exercises	162
9	Independence	167
9.1	Independent Events	168
9.2	Independence for σ -Fields and Random Variables	169
9.3	The Borel-Cantelli Lemma	177
9.4	Construction of Independent Sequences	181

9.5	Sums of Independent Random Variables	182
9.6	Convolution Semigroups	186
9.7	The Poisson Process	188
9.8	Exercises	195
10	Convergence of Random Variables	199
10.1	The Different Notions of Convergence	199
10.2	The Strong Law of Large Numbers	204
10.3	Convergence in Distribution	209
10.4	Two Applications	216
10.4.1	The Convergence of Empirical Measures	216
10.4.2	The Central Limit Theorem	219
10.4.3	The Multidimensional Central Limit Theorem	221
10.5	Exercises	223
11	Conditioning	227
11.1	Discrete Conditioning	227
11.2	The Definition of Conditional Expectation	230
11.2.1	Integrable Random Variables	230
11.2.2	Nonnegative Random Variables	233
11.2.3	The Special Case of Square Integrable Variables	237
11.3	Specific Properties of the Conditional Expectation	238
11.4	Evaluation of Conditional Expectation	242
11.4.1	Discrete Conditioning	242
11.4.2	Random Variables with a Density	242
11.4.3	Gaussian Conditioning	244
11.5	Transition Probabilities and Conditional Distributions	248
11.6	Exercises	251
Part III Stochastic Processes		
12	Theory of Martingales	257
12.1	Definitions and Examples	257
12.2	Stopping Times	263
12.3	Almost Sure Convergence of Martingales	266
12.4	Convergence in L^p When $p > 1$	274
12.5	Uniform Integrability and Martingales	280
12.6	Optional Stopping Theorems	284
12.7	Backward Martingales	290
12.8	Exercises	296
13	Markov Chains	303
13.1	Definitions and First Properties	303
13.2	A Few Examples	308
13.2.1	Independent Random Variables	308
13.2.2	Random Walks on \mathbb{Z}^d	309

13.2.3	Simple Random Walk on a Graph	309
13.2.4	Galton-Watson Branching Processes	310
13.3	The Canonical Markov Chain	311
13.4	The Classification of States	317
13.5	Invariant Measures	326
13.6	Ergodic Theorems	332
13.7	Martingales and Markov Chains	338
13.8	Exercises	343
14	Brownian Motion	349
14.1	Brownian Motion as a Limit of Random Walks	349
14.2	The Construction of Brownian Motion	353
14.3	The Wiener Measure	359
14.4	First Properties of Brownian Motion	361
14.5	The Strong Markov Property	365
14.6	Harmonic Functions and the Dirichlet Problem	372
14.7	Harmonic Functions and Brownian Motion	383
14.8	Exercises	389
A	A Few Facts from Functional Analysis	395
Notes and Suggestions for Further Reading		399
References		401
Index		403