

GRADUATE STUDIES
IN MATHEMATICS **226**

A First Course in Spectral Theory

Milivoje Lukić



Contents

Preface	xiii
Chapter 1. Measure theory	1
§1.1. σ -algebras and monotone classes	1
§1.2. Measures and Carathéodory's theorem	6
§1.3. Borel σ -algebra on the real line and related spaces	10
§1.4. Lebesgue integration	15
§1.5. Lebesgue–Stieltjes measures on \mathbb{R}	24
§1.6. Product measures	30
§1.7. Functions on σ -locally compact spaces	32
§1.8. Regularity of measures	35
§1.9. The Riesz–Markov theorem	38
§1.10. Exercises	41
Chapter 2. Banach spaces	45
§2.1. Norms and Banach spaces	45
§2.2. The Banach space $C(K)$	48
§2.3. L^p spaces	54
§2.4. Bounded linear operators and uniform boundedness	59
§2.5. Weak-* convergence and the separable Banach–Alaoglu theorem	65
§2.6. Banach-space valued integration	68
§2.7. Banach-space valued analytic functions	71
§2.8. Exercises	74

Chapter 3. Hilbert spaces	77
§3.1. Inner products	77
§3.2. Subspaces and orthogonal projections	82
§3.3. Direct sums of Hilbert spaces	88
§3.4. Orthonormal sets and orthonormal bases	91
§3.5. Weak convergence	97
§3.6. Tensor products of Hilbert spaces	100
§3.7. Exercises	104
Chapter 4. Bounded linear operators	107
§4.1. The C^* -algebra of bounded linear operators on \mathcal{H}	107
§4.2. Strong and weak operator convergence	110
§4.3. Invertibility, spectrum, and resolvents	113
§4.4. Polynomials of operators	118
§4.5. Invariant subspaces and direct sums of operators	119
§4.6. Compact operators	122
§4.7. Exercises	125
Chapter 5. Bounded self-adjoint operators	129
§5.1. A first look at self-adjoint operators	130
§5.2. Spectral theorem for compact self-adjoint operators	136
§5.3. Spectral measures	139
§5.4. Spectral theorem on a cyclic subspace	141
§5.5. Multiplication operators	143
§5.6. Spectral theorem on the entire Hilbert space	146
§5.7. Borel functional calculus	149
§5.8. Spectral theorem for unitary operators	153
§5.9. Exercises	155
Chapter 6. Measure decompositions	159
§6.1. Pure point and continuous measures	160
§6.2. Singular and absolutely continuous measures	162
§6.3. Hausdorff measures on \mathbb{R}	169
§6.4. Matrix-valued measures	176
§6.5. Exercises	178
Chapter 7. Herglotz functions	183
§7.1. Möbius transformations	184

§7.2.	Schur functions and convergence	188
§7.3.	Carathéodory functions	190
§7.4.	The Herglotz representation	193
§7.5.	Growth at infinity and tail of the measure	196
§7.6.	Half-plane Poisson kernel and Stieltjes inversion	199
§7.7.	Pointwise boundary values	204
§7.8.	Meromorphic Herglotz functions	210
§7.9.	Exponential Herglotz representation	212
§7.10.	The Phragmén–Lindelöf method and asymptotic expansions	215
§7.11.	Matrix-valued Herglotz functions	216
§7.12.	Weyl matrices and Dirichlet decoupling	219
§7.13.	Exercises	222
 Chapter 8. Unbounded self-adjoint operators		227
§8.1.	Graphs and adjoints	228
§8.2.	Resolvents and self-adjointness	231
§8.3.	Unbounded multiplication operators and direct sums	236
§8.4.	Spectral measures and the spectral theorem	238
§8.5.	Borel functional calculus	243
§8.6.	Absolutely continuous functions and derivatives on intervals	247
§8.7.	Self-adjoint extensions and symplectic forms	253
§8.8.	Exercises	262
 Chapter 9. Consequences of the spectral theorem		267
§9.1.	Maximal spectral measure	268
§9.2.	Spectral projections	270
§9.3.	Spectral type and spectral decompositions	272
§9.4.	Ruelle–Amrein–Georgescu–Enss (RAGE) theorem	275
§9.5.	Essential and discrete spectrum; the min–max principle	278
§9.6.	Spectral multiplicity	283
§9.7.	Stone’s theorem	289
§9.8.	Fourier transform on \mathbb{R}	290
§9.9.	Abstract eigenfunction expansions	293
§9.10.	Exercises	296

Chapter 10. Jacobi matrices	299
§10.1. The canonical spectral measure and Favard's theorem	300
§10.2. Unbounded Jacobi matrices	305
§10.3. Weyl solutions and m -functions	309
§10.4. Transfer matrices and Weyl disks	313
§10.5. Full-line Jacobi matrices	319
§10.6. Eigenfunction expansion for full-line Jacobi matrices	322
§10.7. The Weyl M -matrix	325
§10.8. Subordinacy theory	328
§10.9. A Combes–Thomas estimate and Schnol's theorem	334
§10.10. The periodic discriminant and the Marchenko–Ostrovski map	336
§10.11. Direct spectral theory of periodic Jacobi matrices	347
§10.12. Exercises	352
Chapter 11. One-dimensional Schrödinger operators	359
§11.1. An initial value problem	361
§11.2. Fundamental solutions and transfer matrices	367
§11.3. Schrödinger operators with two regular endpoints	373
§11.4. Endpoint behavior	379
§11.5. Self-adjointness and separated boundary conditions	386
§11.6. Weyl solutions and Green's functions	390
§11.7. Weyl solutions and m -functions	394
§11.8. The half-line eigenfunction expansion	399
§11.9. Weyl disks and applications	407
§11.10. Asymptotic behavior of m -functions	415
§11.11. The local Borg–Marchenko theorem	423
§11.12. Full-line eigenfunction expansions	425
§11.13. Subordinacy theory	429
§11.14. Potentials bounded below in an L^1_{loc} sense	433
§11.15. A Combes–Thomas estimate and Schnol's theorem	439
§11.16. The periodic discriminant and the Marchenko–Ostrovski map	443
§11.17. Direct spectral theory of periodic Schrödinger operators	450
§11.18. Exercises	453

Bibliography	459
Notation Index	467
Index	469